Probabilistic Methods in Model Validation

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Albuquerque, New Mexico

International Modal Analysis Conference

2009

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for the United States Department of Energy under contract DE-AC04-94AL85000.

Outline

- Motivation
- Introduction
- Modeling Univariate sources
- Modeling Bivariate Sources
- Propagation of Uncertainty
- Comparing Predictions to Experiments
- Conclusions

Motivation

Real structural systems behave randomly

- Structural reliability analysis was introduced in the US in the mid 1950s
- Random vibration analysis was introduced in the US in the late 1950s
- To characterize actual structural behavior we require theories of statics and probability
- To model actual structural behavior we require probability models

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Introduction

This talk

- **Introduces the ideas of probability and statistics**
- **Shows how they can be used in engineering modeling and analysis**

Ideas

- **Randomness – basic means for measuring and modeling it**
- **Random experiment, random variable, mean, variance, standard deviation, probability distribution**
- **Introduction in terms of practical example**

Probabilistic UQ

Probability and statistics include tools to

1. **Characterize randomness in experimental data**
2. **Create models to simulate random phenomena**
3. **Compare model predictions to experimental data (Validation)**

First, consider 1 and 2

- **Univariate data**
- **Bivariate data**
- **Propagation of uncertainty (through FEM)**

Then, consider 3

Modeling and Simulation of Univariate Random Source

- Characterize bonding material (shell structure described in Introductory talk)
- A property of interest is shear modulus, *G*
- Experiment to establish probability model
  - Fabricate nominally identical replicates of experimental system
  - Test to infer value of *G* in each sample
  - Perform statistical analysis of data

Experiments

- **Performed to develop probability model for modulus of elasticity, *E*, and shear modulus, *G*, of bonding material**
- **Structure description**
- **Five nominally identical replicates**
- **Modal tests**
- **"Measure" modal frequencies**
- **Create FEM**
- **Compute model frequencies**
- **Vary *E* & *G* to cause model mode frequencies to match experiment mode frequencies**

Diameters = 1.125 *in*
Steel disks = 0.375 *in*
Bond disks = [0.506,0.507,0.507,0.505, 0.506] *in*

Experiments

Structure index	Shear mode frequency (Hz)	Axial mode frequency (Hz)
1	1750	2210
2	1800	2275
3	1787	2240
4	1762	2215
5	1750	2230

Test Index, *i*	**G_i (*psi*)**
1	**3151.9**
2	**3334.5**
3	**3297.9**
4	**3201.8**
5	**3141.2**

Diameters = 1.125 *in*
Steel disks = 0.375 *in*
Bond disks = [0.506,0.507,0.507,0.505, 0.506] *in*

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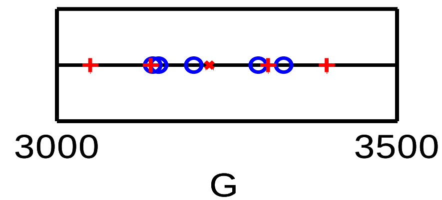
Modeling and Simulation of Univariate Random Source

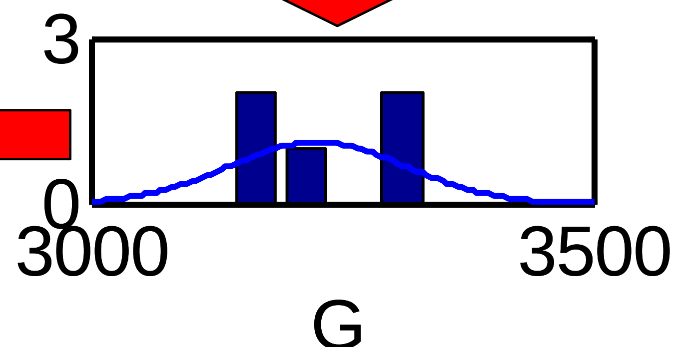
Test Index, *i*	$G_i\,(psi)$
1	3151.9
2	3334.5
3	3297.9
4	3201.8
5	3141.2

Bond material shear modulus

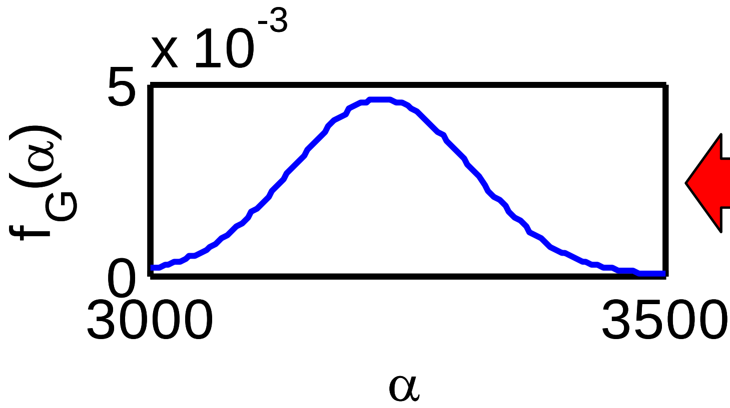
$\bar{x}$	$3225.4\,psi$
s_X^2	$7555.0\,(psi)^2$
s_X	$86.9\,psi$



Estimated moments and number line with data



Histogram and assumed shape of PDF



Best fit normal PDF

Statistical Estimators

- **Sample mean**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- **Sample variance**

$$s_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

- **Sample standard deviation**

$$s_X = \sqrt{s_X^2}$$

- **Histogram leads to probability density function (PDF). Example: Normal PDF**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} exp\left[-\frac{1}{2\sigma_X^2}\left(x - \mu_X\right)^2\right] \qquad -\infty < x < \infty$$

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Statistical Estimators

- PDF describes distribution of realizations of random variable …
- Function must satisfy some requirements to be a PDF

$$f_X(x) \geq 0 \qquad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

- Once the PDF is known (assumed), it can be used to compute probabilities of events

$$P(a < X \leq b) = \int_a^b f_X(x)dx \qquad -\infty < a < b < \infty$$

σ_X^2

Statistical Estimators

- More important (for present purposes) information about distribution can be used to generate realizations (samples) of a random variable
- Example: Normal random variable with mean μ_X, and variance σ_X^2, has realization

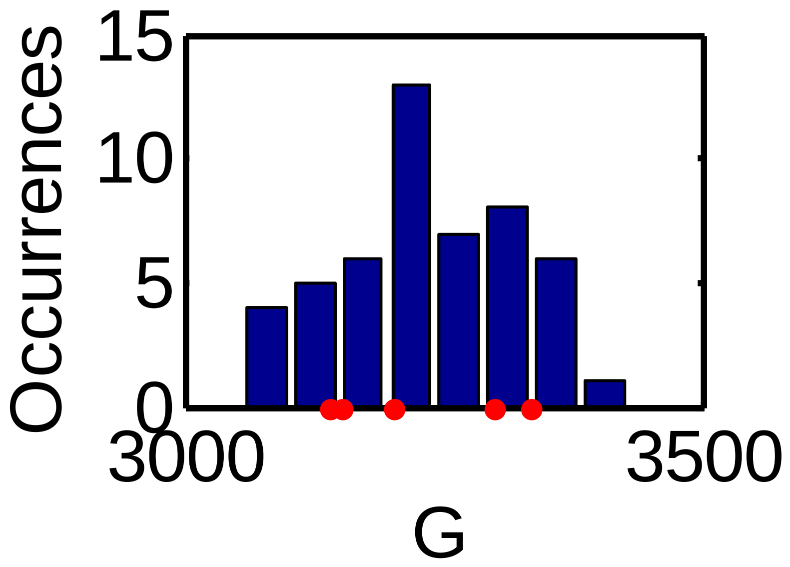
$$x = \mu_X + \sigma_X r$$

  where r is a realization from a standard normal random source (generated in MATLAB, EXCEL, etc.)
- Any number of realizations, x, can be generated

σ_X^2

Generation of Random Data

- **Example: Generate data from a source with mean and variance, $\mu_X = 3225.4\,psi$, $\sigma_X^2 = 7555.0(psi)^2$, the estimated source of the shear moduli, then show the histogram (50 realizations)**



Other distribution models are possible and some are very simple. For example, the lognormal model.

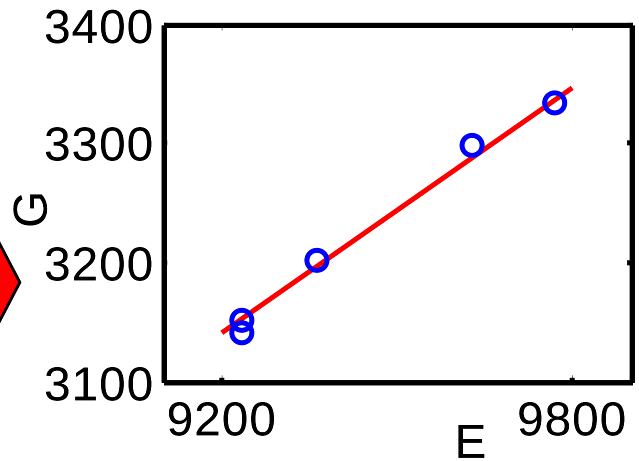
Modeling and Simulation of Bivariate Random Source

- **Wish to characterize bonding material in terms of shear modulus, *G*, and modulus of elasticity, *E*, simultaneously**
- **Use data from experiments described above**

Modeling and Simulation of Bivariate Random Source

Test Index, *i*	G_i (*psi*)	E_i (*psi*)
1	3151.9	9235
2	3334.5	9770
3	3297.9	9630
4	3201.8	9362
5	3141.2	9235

Shear modulus and modulus of elasticity



$\overline{(G-\overline{G})(E-\overline{E})}$	20990
$\hat{\rho}_{GE}$	0.9965

Scatter-plot and sample covariance and correlation coefficient

Statistical Estimators

- **Sample covariance … meaning**

$$\overline{(X-\bar{x})(Y-\bar{y})} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

- **Sample correlation coefficient … meaning**

$$\hat{\rho}_{XY} = \frac{\overline{(X-\bar{x})(Y-\bar{y})}}{s_X s_Y}$$

These statistics characterize the degree of linear relation between a pair of random variables

Statistical Estimators

- **Regression - One way to develop a relation easily used to generate random variable pairs**
- **One form of the formula is**

$$y_i = \hat{\alpha} x_i + \varepsilon_i \qquad i = 1,\ldots,n$$

where $\hat{\alpha}$ **is a constant and** $\varepsilon_i, i = 1,\ldots,n$ **are samples from a zero-mean random source with variance** s^2

- **Formula for** $\hat{\alpha}$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$\hat{\alpha} = 0.3415$

For *G* vs. *E* data

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Statistical Estimators

- **Estimate the variance, s^2, of the source of the $\varepsilon_i, i = 1,\dots,n$, by evaluating the regression formula**

$$y_i^{(mod)} = \hat{\alpha} x_i \qquad i = 1,\dots,n$$

at the sample points $x_i, i = 1,\dots,n$, then use the results in

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - y_i^{(mod)}\right)^2$$

(In this example variance is very small)

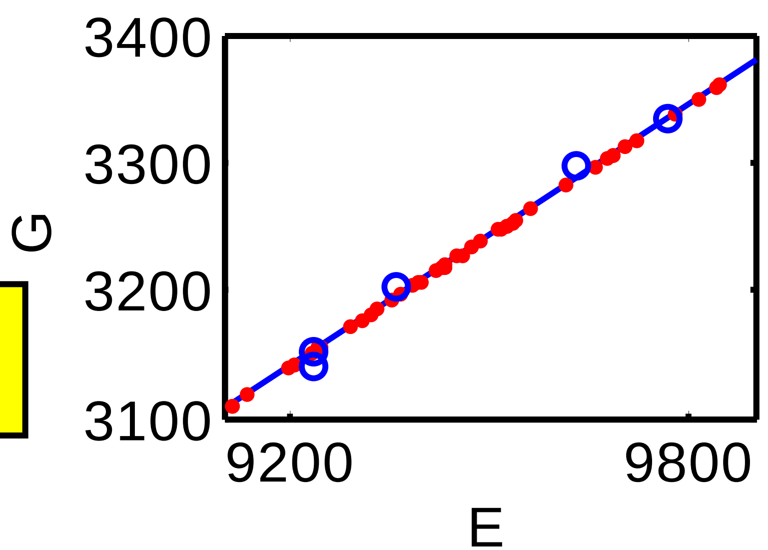
Generation of Bivariate Random Data

- **Find the probability model for the *x* – data, and use it to generate** $x_i, i = 1,\ldots,n$
- **For each generated *x* compute**

$$y_i^{(mod)} = \hat{\alpha} x_i \qquad i = 1,\ldots,n$$

- **Generate *n* zero-mean random variates,** $\varepsilon_i, i = 1,\ldots,n$ **each with variance,** s^2**, and add these to the** $y_i^{(mod)}, i = 1,\ldots,n$

Results for *G* vs. *E* data



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Propagation of Randomness through a Model

- Normally develop deterministic finite element models to approximate system behavior. They map deterministic inputs to deterministic outputs

$$w = g_{FE}(p, E, G)$$

  - w is response measure of interest
  - $g_{FE}(\cdot)$ embodies FEM operations
  - p is input data
  - E, G are random samples of modulus of elasticity and shear modulus

Propagation of Randomness through a Model

- When one or more inputs are random, the Monte Carlo method can be used to characterize system behavior
  - Generate realizations of input random variables
  - Use them, individually, as FEM inputs (along with deterministic inputs) to compute outputs
  - Statistically analyze outputs

Propagation of Randomness through a Model

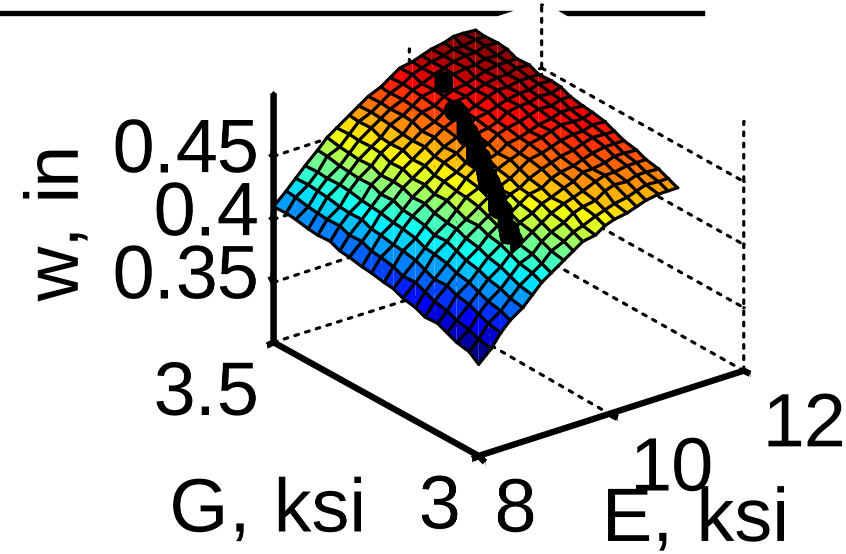
Example:

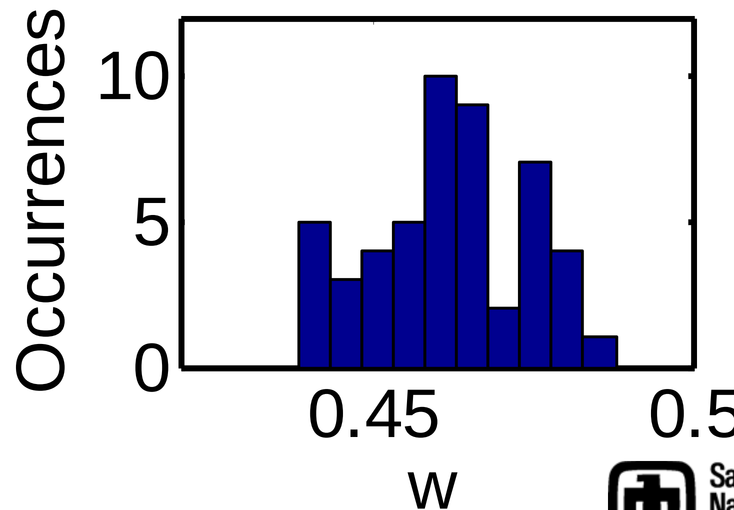
- *w* **is peak displacement response at a point on a structure**
- **Response surface as function of** *E* **and** *G* **shown at right**
- **Responses to generated** (*E*,*G*) **inputs shown as dots**
- **Statistics of response**

$$\bar{w} = 0.4621 in$$

$$s_W^2 = 1.4513 \times 10^{-4} in^2$$

$$s_W = 0.0120 in$$





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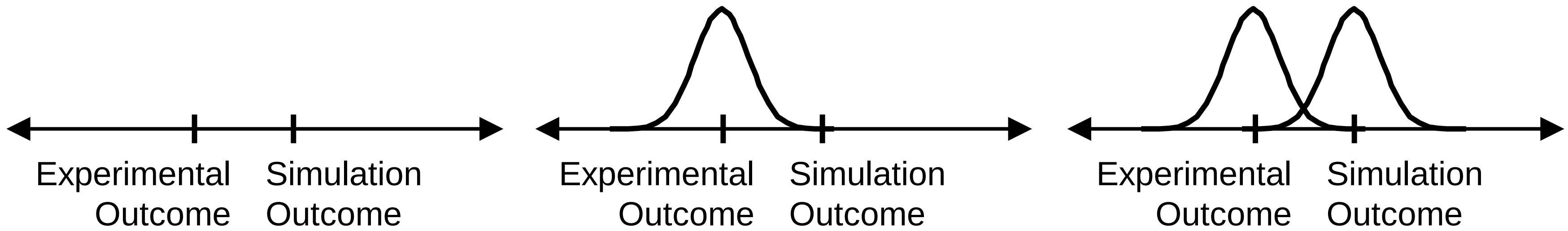
Comparison of Model Predictions to Experimental Responses

Comparison depends on data and models available



Validation Case 1

Deterministic/Deterministic Comparison

Validation Case 2

Deterministic/Random Comparison

Validation Case 3

Random/Random Comparison

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Comparison of Model Predictions to Experimental Responses

Example:

- **Validation case 2. One measured peak displacement response** $w^{(exp)} = 0.4210\,in$
- **Very simple validation criterion requires peak response** $w^{(exp)}$ **to lie within interval**

$$\left[\overline{w} - 1.96 s_w, \overline{w} + 1.96 s_w\right]$$

  where parameters that define interval are from model predictions generated, above.
- **In this case the interval is** [0.4385,0.4857] **– model not valid**
- **Criterion could be made less severe by using**

$$\left[\alpha\left(\overline{w} - 1.96 s_w\right), \beta\left(\overline{w} + 1.96 s_w\right)\right]$$

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Comparison of Model Predictions to Experimental Responses

Example:

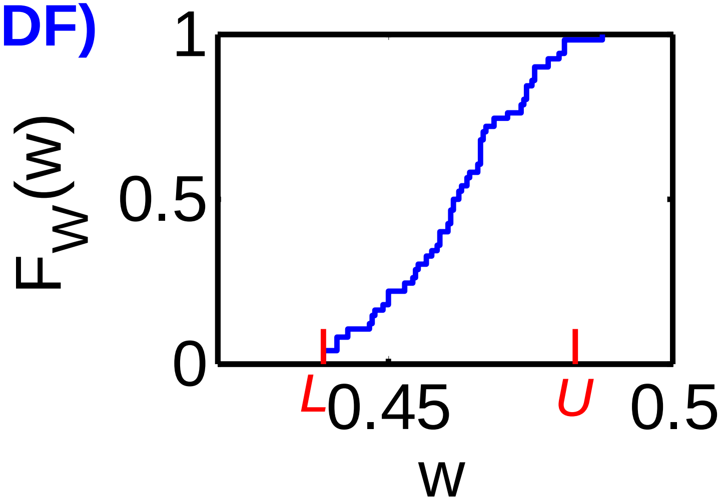
- **Validation case 2. One measured peak displacement response**

$$w^{(exp)} = 0.4210\, in$$

- **Another very simple validation criterion requires $w^{(exp)}$ to lie within $[L,U]$ where these limits come from a cumulative distribution function (CDF)**
- **For 95% probability interval**

$$[L,U] = [0.4387, 0.4829]$$

- **Model not valid**
- **Criterion could be made less severe by using $[\alpha L, \beta U]$**



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Comparison of Model Predictions to Experimental Responses

Example: Two-Sample Kolmogorov-Smirnov Test

- **Validation case 3. Multiple measured peak displacement responses and multiple model-predicted responses**
- **Draw empirical CDFs of measured response data and model-predicted response data**
- **Measure greatest vertical distance, Δ, between two curves**
- **Establish KS criterion, Δ_{KS}, for hypothesis that "data come from a single random source"**
- **Compare Δ to Δ_{KS}. If $\Delta \leq \Delta_{KS}$ then model valid.**

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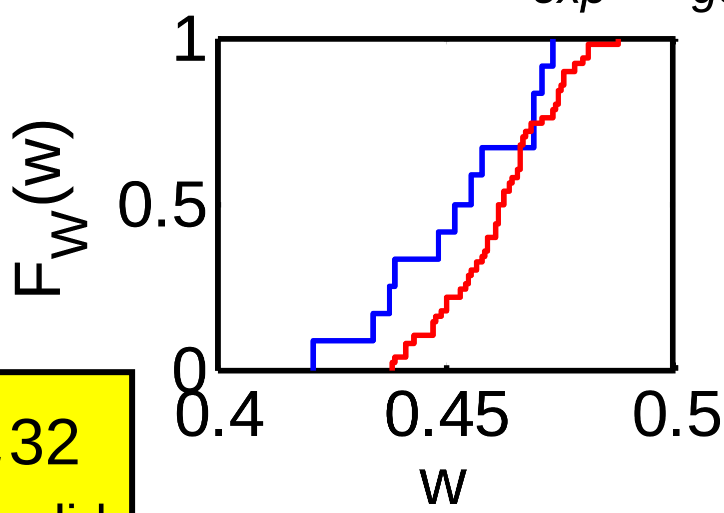
Comparison of Model Predictions to Experimental Responses

Effective sample size: $n_{eff} = \frac{n_{exp} n_{gen}}{n_{exp} + n_{gen}} \cong 10$



$\Delta = 0.32$

Model valid

Empirical CDFs from experimental data (blue) and model predictions (red)

Effective Sample Size	Critical Statistic, Δ
5	0.56
10	0.41
20	0.34
25	0.29
30	0.26
40	0.21
Large *n*	$1.36/\sqrt{n}$

Criteria for Kolmogorov-Smirnov tests with significance level of 0.05

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Conclusions

Several elements of probabilistic analysis & modeling presented

- Models for random, univariate data
- Generate samples from the univariate model
- Models for random bivariate data
- Generate samples from the bivariate model
- Propagate randomness through deterministic finite element model (FEM)
- Compare predictions from FEM to experimental results in two cases
  – One experimental result is available and a probability model is available for the FEM outputs
  – Probability models are available for both the experimental results and the FEM