

Model Correlation and Calibration

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NOMENCLATURE

FEM	Finite Element Model
MAC	Modal Assurance Criterion
S	Matrix of sensitivities of frequencies to parameters
$\Delta\bar{\mathbf{p}}$	Vector of predicted changes in parameters to update the model
$\Delta\bar{\mathbf{f}}$	Vector of differences between test frequencies and model frequencies
$COV\Delta\bar{\mathbf{p}}$	Covariance matrix for parameters
m	Number of frequencies
n	Number of parameters
σ_f^2	Variance of the $\Delta\bar{\mathbf{f}}$ vector calculated as $\Delta\bar{\mathbf{f}}^T \Delta\bar{\mathbf{f}} / (m-n)$

ABSTRACT

Analytic model validation assesses the usefulness of a model for its intended purpose. Validation of the model should be based on a blind prediction of test results, so that the predictive capability is demonstrated. However, there is certainly a place in the validation process for exercising the model in correlation and calibration before making the final blind validation prediction. In many cases, the initial model deviates from a useful state for unknown or unquantified reasons. Then model correlation exercises are performed. The term "correlation" for structural dynamics comes from the initial one to one correlation of the modes of the model with modes from a modal test. But correlation really includes more than just this initial comparison. Correlation exercises can uncover unintended errors or incorrect assumptions and simplifications in the thousands of details important to model development. Calibration, on the other hand, is designed to improve estimates on specific uncertain parameters. Sensitivity analysis is examined as a method for calibration. Sensitivity analysis is an inferential process that has definite limits, and three tools for understanding those limits are provided. In some cases, calibration does not provide sufficient model improvement, and additional correlation exercises are performed after the calibration. Often correlation and calibration are most effective when applied to subsystems of the model to isolate specific errors.

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INTRODUCTION

In the field of structural dynamics, "model validation" is a common phrase. However, the meaning of the phrase is sometimes not so clear. In the 80's, the structural dynamics community often used the words "model validation", "model updating" and "model correlation" interchangeably. Any of these terms meant that the parameters of a finite element model had been adjusted or tuned to produce a "better match" of the eigenvectors and eigenvalues of the model with modal test mode shapes and frequencies. In some circles, perhaps this is still the meaning associated with model validation, but model validation technology is advancing. One popular definition is that "Model validation is the process of determining the degree to which a computer model is an accurate representation of the real world from the perspective of the intended model application [1]". In general, the "accurate representation of the real world" comes from some type of test data. Here we take the approach that the model must make a blind prediction of the test response, and then be compared with the response to see if there is agreement within some predetermined bounds, to decide if the model is valid. The implication of this approach is that the final validation test data are not used for correlation or calibration. However, correlation and calibration is one important step in the validation process. Consider the steps to validation listed below which are given by Urbina, Paez, et al [2].

- Preliminary Steps
 - Specify model use/purpose (what decision is to be made)
 - Specify response measures (what the model predicts)
 - Specify validation features and metrics and comparison domain
 - Specify calibration experiments
 - Specify validation experiments
 - Specify adequacy criteria
- [Perform calibration experiments/Calibrate model parameters](#)
- Validation
 - Perform experiment
 - Make predictions
 - Calculate metrics/compare with adequacy criterion
- Subsequent Action
 - Not valid – Reformulate model/Additional calibration
 - Valid – Make predictions

This paper focuses on the step highlighted in blue. The author proposes that this step is expanded to include correlation as well as calibration. This is the part of the validation process where significant insight is gained from the model by exercising it to predict some test response that is *not* the final validation response. In this step, the analyst and experimentalist should work together to develop insight. Their combined experience and knowledge will provide much greater value than the insight of either one alone. In this step, improvements in representing the physics and the computational implementation of the model are desired. The focus will be on structural dynamics. Assume that a modal test has been performed, and a model has made predictions for the modal test results. The best approach for correlation and calibration is to break the structure down to subassemblies and to use modal tests and model predictions at the subassembly level. In so doing, uncertainties associated with interfaces and boundary conditions are better understood and effects of uncertain parameters are isolated. The correlation and calibration efforts for one model subassembly and the associated modal test are described later. Assume that the model and modal test produce a set of target modal frequencies and mode shapes that are independent in the mode shape matrix. It is also assumed here that the test frequencies and shapes have been accurately extracted.

CORRELATION - Gain Physical and Modeling Insight by Observing Differences between Model and Test

One of the valuable results from a model prediction and modal test is the physical insight that may be provided by comparing the mode shapes and frequencies. First determine which frequencies have the largest percentage difference and compare the mode shapes by eye. Sometimes this may guide the analyst and experimentalist to the physical area of the model where there may be differences. Where there are large frequency differences, the mass or stiffness has to be in error. Analysis of shape differences may provide insight to where the differences

may be located. At this point various sorts of error should be considered, not just errors in certain parameter values such as a modulus. The frequency and mode shape comparisons can be evidence of:

1. Faulty element connectivity in the FEM (evidenced by extreme local displacements in the FEM mode shapes);
2. Faulty boundary conditions in the FEM or possibly the modal test (evidenced by mode shapes that appear unduly constrained);
3. Lack of convergence of the FEM due to insufficient element discretization;
4. Errors or oversimplifications in model form, element choice or defeaturing;
5. Typographical errors such as providing the modulus of steel instead of aluminum or missing a decimal place;
6. Possible errors in parameters that affect the mass and stiffness (addressed more in the calibration section).

In past work, all these types of errors have been discovered in correlation activities. Through these observations, both analyst and experimentalist gain insight into the hardware simply by observing the way the structure is exercised in its modes of vibration. Sometimes it becomes obvious that a particular physical area of the structure is very important to certain modes of vibration, and the analyst may focus on refining that area of the model. Testing insight can also be gained. For example, in one modal test of a system on a seismic mass and air springs, one of the air springs had leaked, and the mass was resting on a stop at one corner, providing an improper boundary condition. This condition was recognized by observing the mode shapes. For a complex system model, more insight is gained by correlating individual subassemblies because this isolates and reduces the number of physical interactions. Focus on non-parametric errors 1-5, and at the end of the correlation process, consider which parameters are most likely to be in error to prepare for the calibration effort.

CALIBRATION - Gain Insight from Matrix of Sensitivities of Frequencies to Parameters

After the non-parametric errors have been corrected, the calibration of specific parameters may be performed. Many FE codes can automatically calculate the sensitivity of the modal frequencies (and other quantities) to various parameters chosen by the analyst. The analyst and experimentalist can work together to select parameters that are uncertain in the model. For example, the modulus of a composite material may not be fully characterized from lab tests, so that it has significant uncertainty. After a list of uncertain parameters is compiled, the analyst utilizes the FE code to calculate the sensitivity matrix for target modes. If this is not a standard capability of the code, a simple finite difference method can be used to calculate the sensitivity matrix by performing $n+1$ runs of the finite element code for n parameters of interest. Normalize the matrix so that one can see what percentage change in a parameter produces a certain percentage change in frequency. Sensitivity methods have been used for a long time to identify parameters[3]. An estimate of the changes in parameters may be obtained to minimize the frequency errors from

$$\Delta \bar{\mathbf{f}} = \mathbf{S} \Delta \bar{\mathbf{p}} \quad (1)$$

where \mathbf{S} is the sensitivity matrix, $\Delta \bar{\mathbf{p}}$ is the vector of parameter changes required to produce the frequency changes in vector $\Delta \bar{\mathbf{f}}$. The $\Delta \bar{\mathbf{f}}$ vector is the frequencies of the model subtracted from the frequencies of the test. The sensitivity matrix gives the corresponding change in frequency associated with a small change in a parameter. Each row is associated with a modal frequency and each column is associated with a single parameter. Each sensitivity value is usually normalized by the initial estimate of the parameter value and the initial FE predicted frequency and so has the form $(\Delta f \times p)/(f \times \Delta p)$. Weighting matrices may be added to this equation[4], but for now let us use equation (1) directly to learn what the model can tell us about the parameters that have been chosen. So that statistical analysis may be utilized, equation (1) should be over determined, that is, the number of frequencies should be greater than the number of parameters. The change in parameters may be calculated in a least squares sense by multiplying the pseudoinverse of the sensitivity matrix times the $\Delta \bar{\mathbf{f}}$ vector. Equation (1) is based on a first order Taylor series expansion which assumes that the model is already "close" to the solution. Because this may not be true, this process may be iterative. Now let us consider some of the possibilities for the set of parameters that have been chosen.

1. *None of the parameters are in error, but the model form is wrong.* If this is the case, equation (1) may calculate changes in the parameters to match the frequencies that are really not valid, but are correcting for other errors in the model. Always attempt to use the correlation to identify the non-parametric errors so that effort is not wasted in worthless calibration.
2. *Some parameters are in error in the model, but they have not been included in the parameter list.*
3. *Some of the parameters in the list are in error, but others that are not in the list are in error.*

4. *All of the important parameters are in the list as well as some that are NOT in error.* If the parameter list falls into categories 2,3 and 4 one would like to have some way of knowing which parameters should NOT be adjusted.
5. *Only parameters that are in error are in the list.* If this is the case, one would assume that equation (1) may be solved directly to calibrate these parameters. However, there are cases where this is not true, and only a subset of the parameters can be adjusted with any validity. If the modal frequencies are not sensitive to one of the parameters, that parameter cannot be adjusted with any confidence by equation (1).

Three tools described below help the analyst know the limits of sensitivity analysis. An example problem is provided here to show the value of these tools. A simple conical shell with three layers of material is modeled with NASTRAN. Each material is isotropic and its constitutive behavior is described by the modulus of elasticity and Poisson's ratio. There are only six parameters for this model. Initial starting parameters are defined for the FEM and another model representing the modal test result (which is defined as truth) has different values for four of these parameters. Figure 1 shows the FEM with the three layers and Figure 2 plots of two of the mode shapes compared to the undeformed mesh.

Table I shows the initial FE model frequencies and the modal test frequencies.

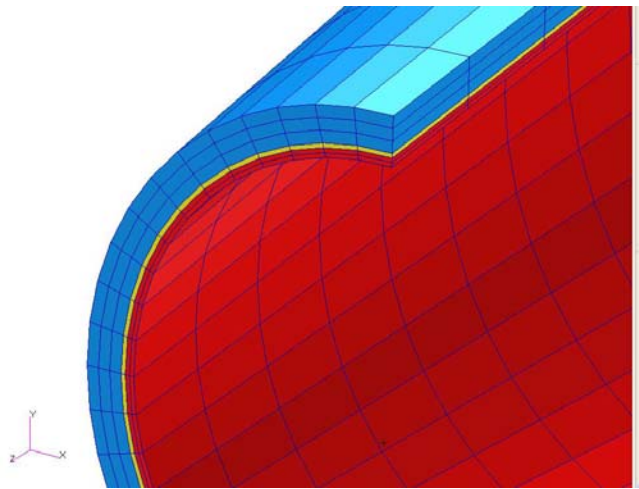


Figure 1 - FEM of Three Layer Cone Structure

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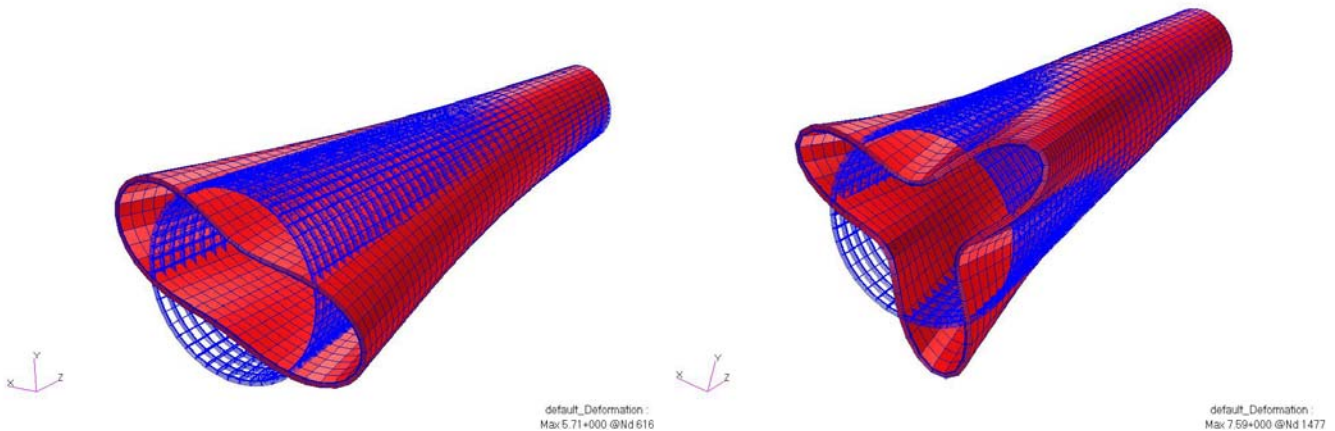


Figure 2 - Ovaling Modes of the Shell Structure

Table 1 - Initial Model Frequencies Vs Modal Test

FE Frequency	Modal Test Frequency	% Error
124.2	149.3	-16.8%
268.1	318.6	-15.9%
277.3	342.6	-19.1%
442	537.6	-17.8%
461.1	574.8	-19.8%
470.7	559.6	-15.9%
641.3	675.9	-5.1%

The sensitivity matrix for seven frequencies and six parameters was calculated and the resulting parameter changes were calculated by multiplying the pseudoinverse of the sensitivity matrix times the frequency error vector. After introducing the parameter changes for all six parameters, theoretically, this reduced the frequency error to less than 0.1 percent for every frequency. Consider the initial and final errors of the parameters compared to the true parameter value in Table 2.

Table 2 - Parameter Errors Before and After "Blind" Calibration of All Six Parameters

Parameter	Initial % Error	% Error after Calibration
E outer	-20.0000	26.5776
nu outer	0	-35.0651
E mid	-65.0000	-39.7927
nu mid	-33.3333	-34.9270
E inner	2.0408	-33.8752
nu inner	0	-68.4929

Although the frequency error has been reduced to near zero by adjusting all six parameters, the error for most of the parameters is greater than it was in the initial estimate! How can this be? There is more than one cause.

Sensitivity Matrix Rank - Tool 1

A quick way to get some insight on the maximum number of parameters one can actually calibrate is to perform a singular value decomposition of the sensitivity matrix and plot the singular values. Although the matrix may be full rank based on a computational calculation, practically, one cannot adjust with confidence more parameters than there are singular values that are above (about) five percent of the maximum singular value (this is a rule of thumb used by the author). Consider the singular value plot for the shell in Figure 3. There are only two singular values above about 0.05, so that is probably the maximum number of parameters one could effectively calibrate with this sensitivity matrix. Even though there are seven frequencies to provide information about six parameters, Figure 3 indicates that there are really only two strong pieces of information in the sensitivity matrix. This could mean that there are only two vectors that have strong amplitude (in which the sensitivity to a parameter is readily apparent). Note that the sensitivity matrix does not give any information about parameter errors. It only provides guidance on the maximum number of parameters one could possibly correct.

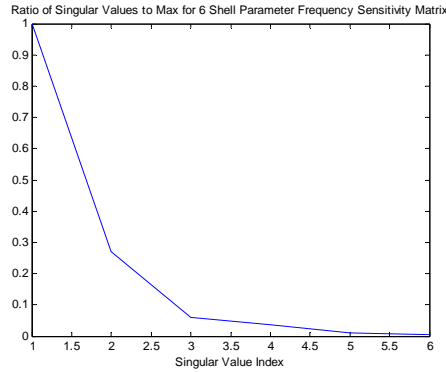


Figure 3 - Singular Values for the Shell Sensitivity Matrix

Parameter Correlation - Tool 2

Another reason for a small number of large singular values associated with the sensitivity matrix is that certain columns of the sensitivity matrix may be nearly parallel to each other (not independent vectors). For parameters with nearly parallel sensitivity vectors, this means that any single one of those parameters could be used to obtain a better match with the test frequencies. The parameters are "correlated", and if they are all included in equation (1), they will all be adjusted, whether they need to or not. If the parameters are correlated, then one cannot tell which parameter is truly in error, or if they may all be in error. In one calibration experience, four different parameters were found to be correlated. Material tests were performed on materials associated with three of the four parameters. It was found that one of the parameters (Young's modulus) was off by a factor of ten from published values! In the initial sensitivity matrix, the frequencies were least sensitive to this parameter of the four correlated parameters, but it was the one which was truly in error. Sensitivity analysis helped point to this group of correlated parameters but could never have identified that one of the parameters had such a large error. Separate material tests were required to make this determination.

The correlation of the sensitivity vectors from two parameters is calculated by the Pearson product-moment correlation coefficient as

$$r_{ij} = \frac{(S_i - \bar{s}_i)^T (S_j - \bar{s}_j)}{\sqrt{(S_i - \bar{s}_i)^T (S_i - \bar{s}_i)(S_j - \bar{s}_j)^T (S_j - \bar{s}_j)}} \quad (2)$$

where S_i is the frequency sensitivity column vector for the i^{th} parameter, and \bar{s}_i is the mean value of the i^{th} sensitivity vector, and the superscript T indicates the transpose of the vector. For those familiar with the modal assurance criterion (MAC) calculation, this is similar to the unsquared MAC with a sign. The analyst should keep track of which parameters have highly correlated frequency sensitivity vectors (absolute value of r_{ij} near one), and realize that equation (1) cannot distinguish between errors in highly correlated parameters.

Table 3 lists the correlation between the six frequency sensitivity vectors associated with each parameter. A numerical value shows the correlation of the frequency sensitivity vector with respect to the parameter at the top of the column to the frequency sensitivity vector with respect to the parameter listed on that row. The symmetric values are left blank.

Table 3 - Correlation of Frequency Sensitivity Among Parameters

	E outer	nu outer	E mid	nu mid	E inner	nu inner
E outer	1	.63	.51	-.55	-.67	-.06
nu outer		1	.60	-.54	-.67	-.38
E mid			1	-.97	-.98	.46
nu mid				1	.96	-.45
E inner					1	-.39
nu inner						1

First note that there is only one correlation coefficient below 0.1. This indicates that all these parameters are somewhat correlated in how they affect the frequencies. Three parameters are HIGHLY correlated with absolute values above 0.95, E mid, nu mid and E inner. The sensitivity matrix will not be able to determine which of these three parameters are in error, and may adjust all three with unrealistic changes (as in Table 4 with 4 parameter changes as explained later). The analyst will be fooled if attempts are made to calibrate all the correlated parameters. A choice has to be made as to which parameters will be included in the calibration. Now, the reasons for the strange results in Table 2 from adjusting all the parameters have become evident. There is only enough power in the sensitivity matrix to adjust two parameters, and three of the parameters are highly correlated. Therefore, adjusting all six parameters gave poor estimates.

Statistically Significant Parameters - Tool 3

Now how does one choose which parameters should really be included in equation (1)? Perhaps there are many parameters the analyst is curious about, but the SVD of the sensitivity matrix indicates that only a few can be identified. Which ones should be chosen? One would like to include the parameters that satisfy two criteria, namely, the frequencies are sensitive to the parameter, and the parameter is in error. One method to determine which parameters are in error and sensitive is by a statistical test of significance. To determine the sensitivity of each parameter to the frequencies, one may determine the standard deviation on the parameters based on equation (1). Branham [5] presents a cogent derivation of statistical analysis for overdetermined least squares solutions associated with equation (1) in chapter 5. First estimate the variance on the frequencies from the left hand side of equation (1) as

$$\sigma_f^2 = \Delta \bar{\mathbf{f}}^T \Delta \bar{\mathbf{f}} / (m - n) \quad (3)$$

where the denominator gets reduced from the total number of frequencies m by the number of parameters n that one is attempting to estimate. The frequency variance can be propagated back through equation (1) to estimate the associated covariance matrix of the parameters as

$$COV \Delta \bar{\mathbf{p}} = \sigma_f^2 [S^T S]^{-1} \quad (4),$$

and the standard error of the mean for each parameter is found by taking the square root of each diagonal element of the covariance and dividing by the number of frequencies as

$$\bar{\sigma}_{mean}(\Delta \mathbf{p}) = \text{sqrt}(\text{diag}(COV \Delta \bar{\mathbf{p}}) / m) \quad (5).$$

The standard error of the mean varies inversely with the sensitivity of the frequencies to parameter. A large standard error of the mean indicates a large change in that parameter will not produce much change in frequency, so it is not very important. Then calculate the z score for parameter i by solving equation (1) to get the estimated change in parameters and dividing by the standard error of the mean for that parameter as

$$z_i = \Delta p_i / \bar{\sigma}_{mean}(p_i) \quad (6).$$

Now what does the z score mean? z_i relates the required parameter change to the noise level of the parameter.

To explain this, one first assumes that the model is "perfect". For a perfect model, the errors in $\Delta \bar{\mathbf{f}}$ would just be experimental errors or "noise" in the estimation of the frequencies. If experimental errors are Gaussian distributed and are propagated to give the standard error on the mean of parameter adjustments in equation (5), one would expect that the correction, Δp_i would be within two standard deviations of the initial estimate with 95 percent confidence. If the z score is greater than two, this indicates that the model was really NOT converged to the "perfect" solution for this parameter. Another way of stating this is that the parameter appears statistically to be in error beyond effects that could be related simply to random noise. In practice, this author does not recommend calculating the z scores with all possible parameters at once, because correlation of parameters can dilute the z score. Use an iterative method to find the most important parameters in the following way:

1. Using only one parameter at a time (that is, use only one column of the sensitivity matrix), calculate the z score for each parameter using equations 3-6 above. The parameter with the largest z score is considered the most important parameter. The large z score indicates the parameter that had the most *statistically significant* change with this test.
2. Retaining the most important parameter, add each other parameter, one at a time (that is, use two columns of the sensitivity matrix), and calculate the z score for all the remaining parameters with equations 3-6. Find the next parameter with the largest z score and retain it as the second most important parameter.
3. Continue adding parameters one at a time until the parameters are prioritized.

4. Plot the z score of each added parameter. If there are many frequencies and the z score is around two or above, then a parameter change is statistically significant. If there are not very many frequencies, the z score may need to be higher to be significant, but the prioritization is the first issue. If the z scores for added parameters drop significantly below two, then the uncertainty of the calibration of the parameters may become large. Consider the z score plot for the conical shell example in Figure 4.

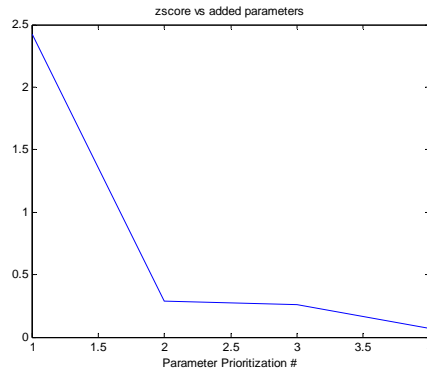


Figure 4- Z Score Vs Added Parameters for Shell Example

Figure 4 shows that even a second parameter added to the problem does not appear to be statistically significant. The parameter selected as most important by this analysis is E mid. In the center column of Table 2 this parameter is the one with the largest initial error. It turns out to be the second most sensitive parameter, but since it has such a large error, it becomes the one most statistically significant when the z score is calculated. By these analyses, the model should be updated by changing this single parameter in the first iteration. Then, in a second iteration, it may be possible to calibrate an additional parameter. Consider the predicted frequency error changes as parameters are added in Figure 5. The standard deviation frequency error is drastically reduced by the addition of one parameter, but only mildly improved with additional parameters. The tools demonstrated in Figures 3, 4 and 5 will help the analyst know the limits of the sensitivity analysis.

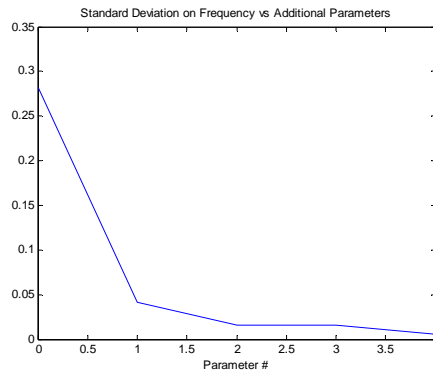


Figure 5 - Predicted Change in Frequency Error as a Function of added Parameters

Now consider the parameter changes for one, two, three and four prioritized parameters vs. the true errors in the parameters as given in

Table 4. (Here the percentage changes are related to the initial FE parameter value, instead of the truth value). When only one parameter was considered there was an improvement in the parameter estimate, although the parameter had such a large error that the linear extrapolation was not enough to improve it to its proper value. With two parameters, two improved, although the one with the most error was not improved as much as with just a single parameter estimate. For this case calibrating more than two parameters causes at least one of the parameters to move away from its true value, rather than toward it.

Table 4- Estimated Parameter Change Percentage As A Function of Added Parameters

Parameter	1	2	3	4	True
E outer		25	26	58	25
nu outer					0
E mid	148	101	74	51	186
nu mid			-145	-51	50
E inner				-33	-2
nu inner					0

Once the significant parameter changes are estimated, they are put into the FE model and a new run is made. If the frequencies do not converge to predicted values in Figure 5, the model was too far from the true solution for the linear sensitivity matrix to be accurate. Additional iterations may be necessary, and as the estimates are improved, another parameter may become statistically significant that was not visible in the first iteration.

Analyst Judgment and Weighting Matrices

After predicting parameter changes, the analyst and experimentalist must check the changes to see if they are reasonable. In the shell example, the fifth and sixth parameters are the modulus and Poisson's ratio for a common material, which are well known to within a few percent. If one were so naive as to utilize the four parameter change in

Table 4, the sensitivity analysis is clearly overcorrecting the modulus of this well known material by an unreasonable amount of 33 percent to compensate for some other error. Weighting matrices[4] are often added to the frequency and parameter vectors in equations (1-6) above to emphasize more important frequencies and to limit changes to realistic values for certain well known parameters. However, they add no insight to the model calibration. If reasonable changes in the parameters do not achieve acceptable calibration, deeper investigation of the model or test will be required to resolve the inconsistency. The analyst and experimentalist should also attempt to define "good enough" before the calibration so that no time is wasted on calibration that offers only diminishing returns. In the shell example, if all other parameters were correct and nu mid was in error with the initial estimate that was 33 percent too low, the frequency standard deviation would be less than one percent. For this problem, nu mid is not a very important parameter, so not much effort should be expended on its calibration.

CONCLUSIONS

The validation process should involve a blind prediction of test responses by a model. However, in the validation process, often subassemblies are modeled and modal tests performed for which the model can be exercised. Correlation at the subassembly level to debug modeling problems, and calibration to estimate uncertain parameters is valuable. In this paper, methods to effectively use sensitivity matrices of modal frequencies to physical parameter values are presented. Some of the limits and pitfalls of the sensitivity approach are presented along with methods to appropriately limit the number of parameters that will be calibrated. The confounding of the estimates when sensitivity vectors of different parameters are correlated is demonstrated. Close cooperation of the analyst and experimentalist are encouraged to maximize the insight gained from exercising of the model.

ACKNOWLEDGMENTS

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