Probability and Statistics in Structural Dynamics

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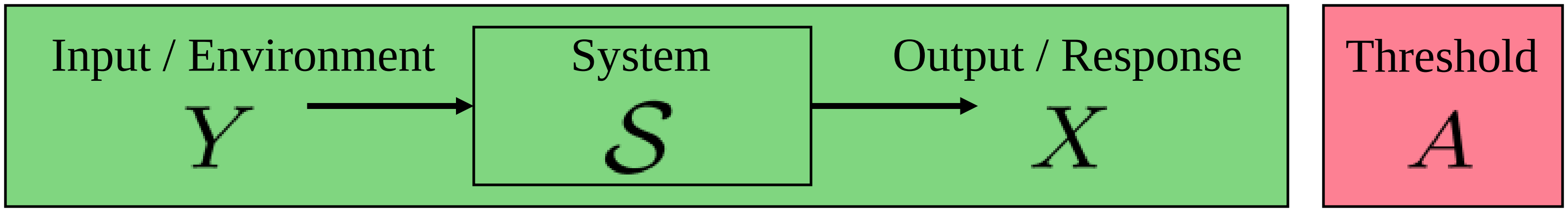
Outline

- Examples of uncertainty in complex systems
- Sensitivity analysis (SA) vs. Uncertainty analysis (UA)
- Probability theory
- Example of deriving distributions for a material
- Methods and tools for uncertainty propagation
- Margin Assessment with Uncertainty (QMU)

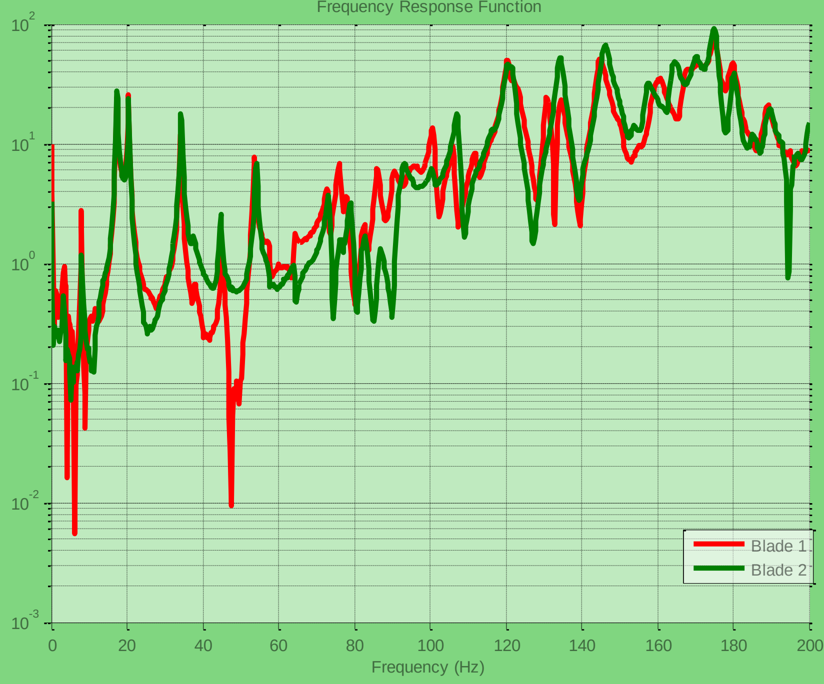
Uncertainty in complex systems



- Uncertainty in some features of ***Y***, $\mathcal{S}$, and/or ***A***
  - Sources include: inherent variability and ignorance
  - Present in: model predictions and experimental observations
- Probabilistic models are one way to quantify the effects of uncertainty on output properties
  - Based on data, theory, and/or expert opinion
  - Calibration / validation of probabilistic models is possible (and necessary)
- Probabilistic representation for input and/or system description implies probabilistic representation for output
- Result: predictions of system performance with quantified uncertainties

Example of unit-to-unit variability

Response FRFs of two identical units



Sensitivity analysis vs. uncertainty analysis

- They are different
- Sensitivity Analysis (SA):
  - How do my code outputs vary due to changes in my code inputs?
  - Local sensitivity: code output gradient data for a specific set of code input parameter values
  - Global sensitivity: the general trends of the code outputs over the full range of code input parameter values
- Uncertainty Analysis (UA):
  - What are the probability distributions on my code outputs, given the probability distributions on my code inputs?
- It may be possible to do SA and UA simultaneously
- If there is uncertainty in a parameter and output is sensitive to the value of that parameter, then UA is needed

Random variables: definitions

- Let X be a random variable
  - Cumulative distribution function (CDF)

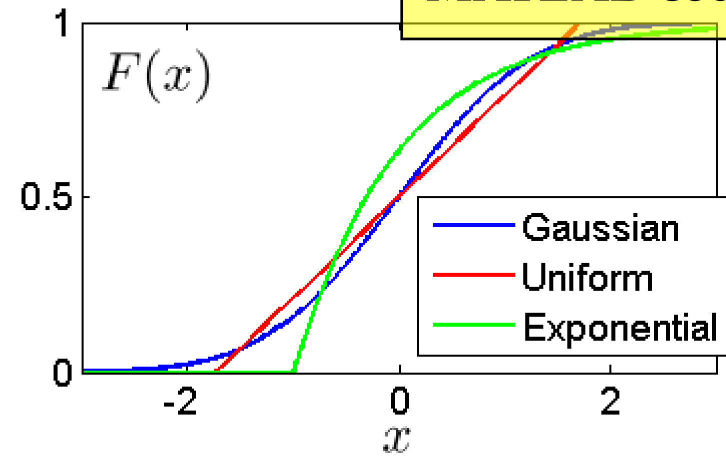
$$F(x) = \Pr(X \leq x), \ -\infty < x < \infty$$

  - Probability density function (PDF)

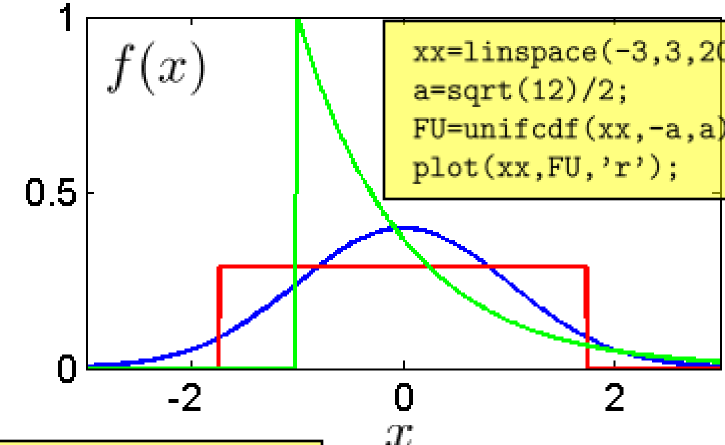
$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$$

- The 3 CDFs/PDFs of X shown here have zero mean and unit variance

MATLAB code




```
xx=linspace(-3,3,200);
a=sqrt(12)/2;
FU=unifcdf(xx,-a,a);
plot(xx,FU,'r');
```




```
fU=unifpdf(xx,-a,a);
plot(xx,fU,'r');
```

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Some properties of random variables: statistical moments

- Absolute moments of X

$$\mathrm{E}[X^p] = \int_{-\infty}^{\infty} x^p\, f(x)\, \mathrm{d}x, \quad p \geq 1$$

$p = 1$ gives the mean of X, denoted by μ

- Central moments of X

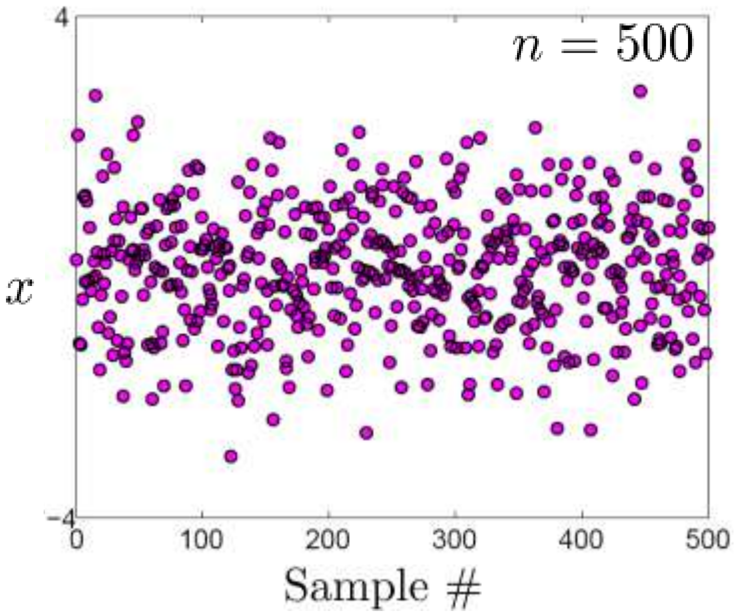
$$\mathrm{E}[(X-\mu)^p] = \int_{-\infty}^{\infty} (x-\mu)^p\, f(x)\, \mathrm{d}x$$

$p = 2$ gives the variance of X, denoted by σ^2

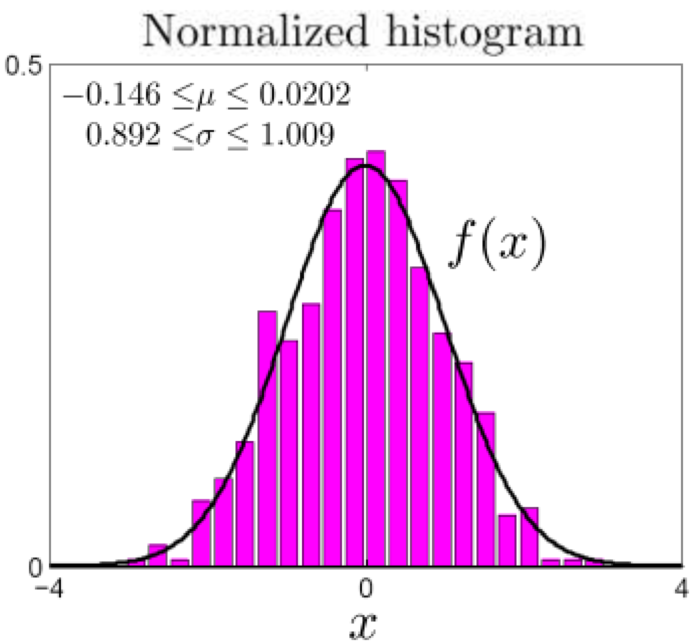
- Comments
  - For applications, we often only know some of the moments (statistics) of X
  - X is only partially defined by its moments (a complete definition requires the PDF or CDF)

Random variables: sample generation

n independent samples of Gaussian random variable *X* with $\mu = 0$ and $\sigma = 1$ by Monte Carlo simulation




```
x=randn(500,1);
plot([1:500],x);
```




```
hist(x,25);hold
xx=linspace(-4,4,250);
f=normpdf(xx,0,1);
plot(xx,f)
```

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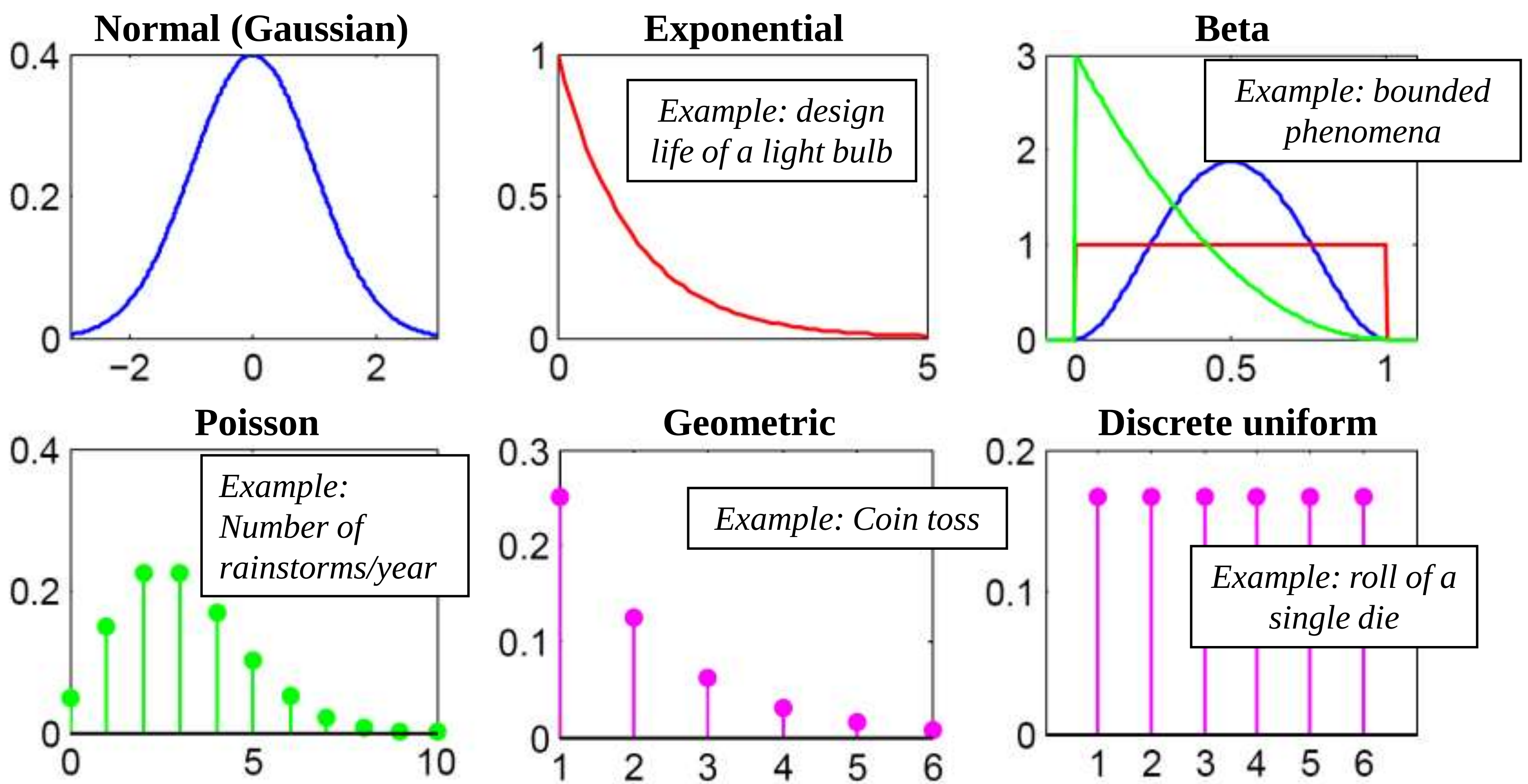
Some commonly used PDFs



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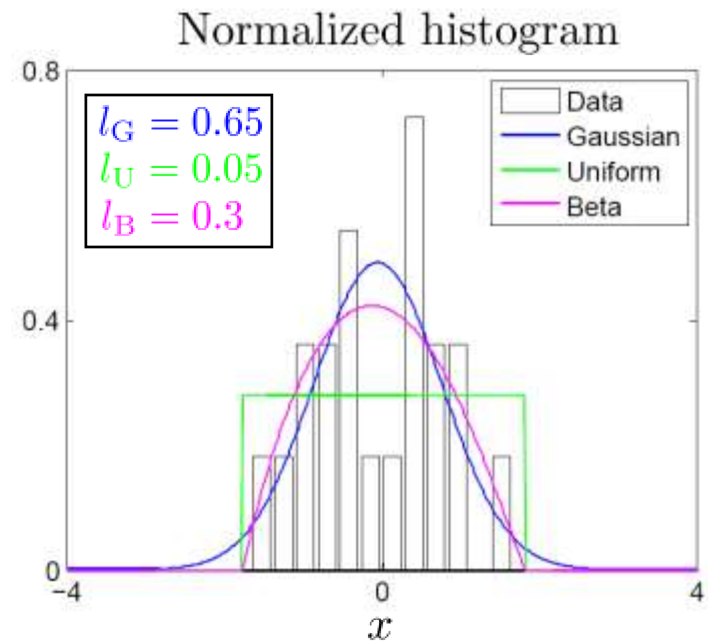
Random variables: modeling techniques

- 20 independent samples of X
  - $x_1, x_2, \ldots, x_{20}$
- Statistical considerations
  - Method of moments
  - Method of maximum likelihood
  - Bayesian Techniques
- Physical considerations
  - Is X continuous or discrete?
  - Is X bounded?
  - Is PDF of X symmetric about $x = 0$?
- Other considerations
  - Conservatism / ease of use
  - What are the consequences / tradeoffs between the different models?




```
[thN1,thN2]=normfit(x);
lG=prod(normpdf(x,thN1,thN2));
```

Tools: Matlab, Minitab, Excel, JMP

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Two or more random variables: random vectors

- Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector
  - Joint CDF

$$F(x_1, x_2) = \Pr(X_1 \le x_1 \cap X_2 \le x_2)$$

  - Joint PDF

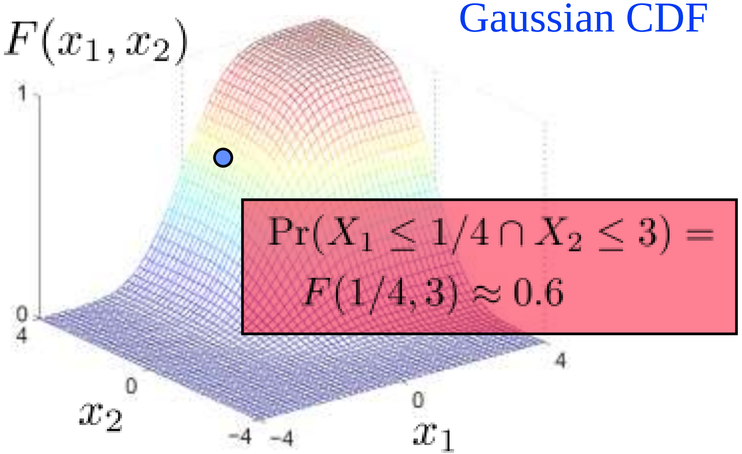
$$f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \, \partial x_2} F(x_1, x_2)$$

  - Special case: X_1 and X_2 are independent (this is a very strong condition)

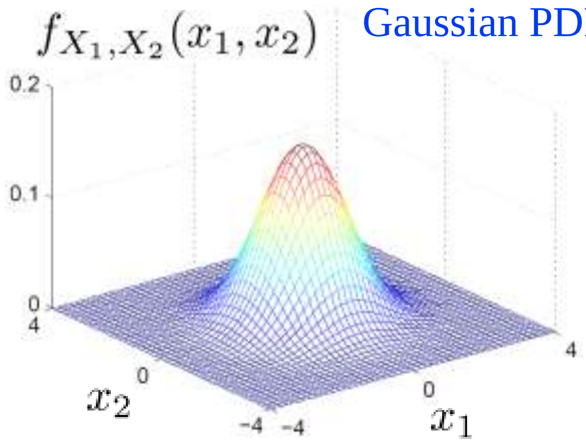
$\Pr(X_1|X_2)=\Pr(X_1)$ $\Pr(X_2|X_1)=\Pr(X_2)$

$$\Pr(X_1 \le 1/4 \cap X_2 \le 3) = \Pr(X_1 \le 1/4) \cdot \Pr(X_2 \le 3)$$

Gaussian CDF



Gaussian PDF



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Random vectors: correlation / covariance

- Correlation of $\boldsymbol{X} = (X_1, X_2)$

$$\mathrm{Corr}(X_1, X_2) = \mathrm{E}[X_1\, X_2]$$

- Covariance of $\boldsymbol{X} = (X_1, X_2)$

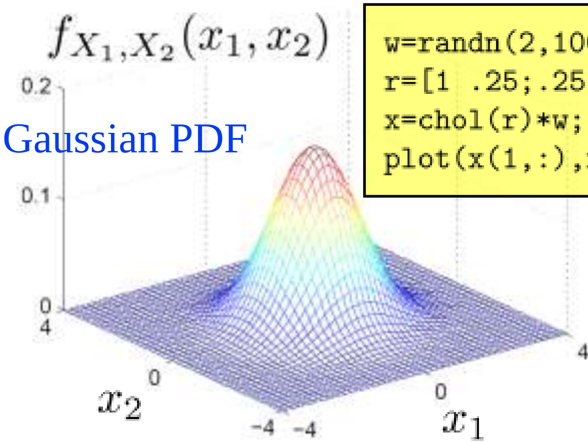
$$\mathrm{Cov}(X_1, X_2) = \mathrm{E}[(X_1 - \mu_1)\,(X_2 - \mu_2)]$$

$$\rho = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Cov}(X_1, X_1) \cdot \mathrm{Cov}(X_2, X_2)}}$$

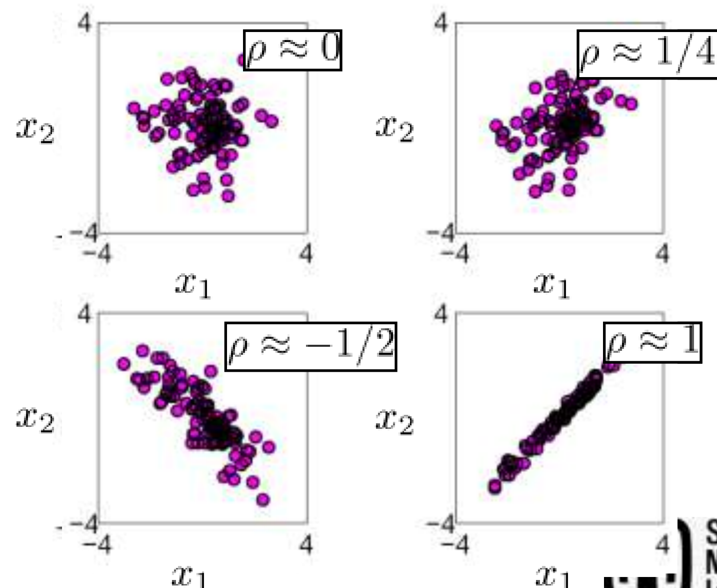
- Special case: X_1 and X_2 are uncorrelated

$$\mathrm{Cov}(X_1, X_2) = 0$$

This does not mean X_1 and X_2 are independent!




```
w=randn(2,100);
r=[1 .25;.25 1];
x=chol(r)*w;
plot(x(1,:),x(2,:),'o')
```



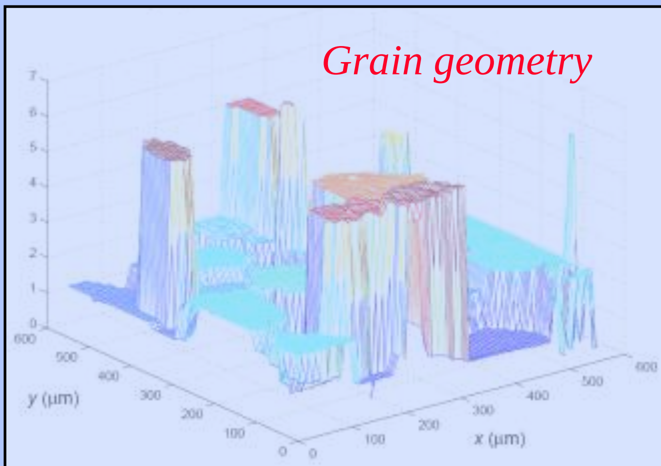
First misconception of uncertainty analysis: "Nature is Gaussian"

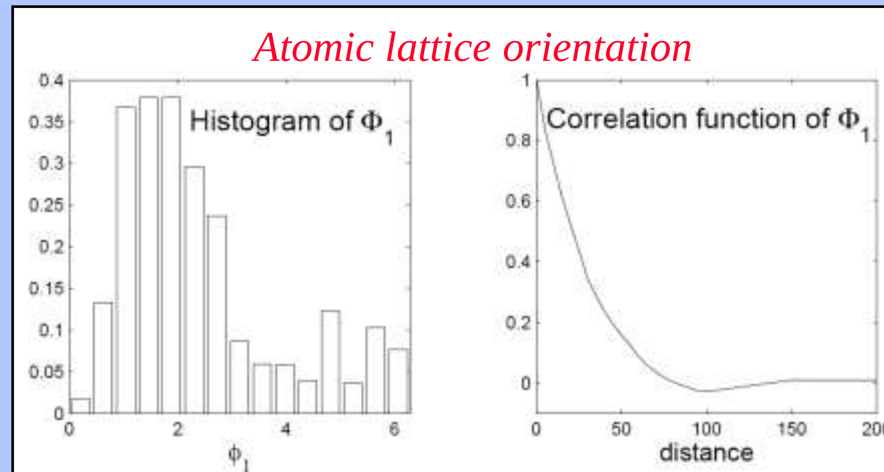
- Analogous to "Nature is Linear"
- Most natural phenomena is bounded and not symmetric
- Examples[1] of non-Gaussian behavior are everywhere!
  - Agriculture (plant size, grain yield)
  - Astronomy (sunspot activity, earth rotation)
  - Economy (unemployment, electricity demand)
  - Environment (wind/pressure forces, ocean wave elevation, seismic ground motion)
  - Material properties (elastic modulus, fatigue crack growth)
  - Medicine (patient survival time, glucose intolerance)
  - Hydrology (precipitation, river flow)
  - Transportation (road roughness, ambulance arrival times)

[1]Examples taken from: M. Grigoriu, *Applied Non-Gaussian Processes*, Prentice Hall, 1995.

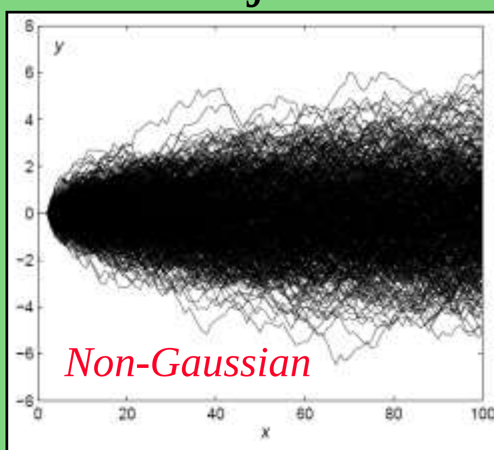
Example of non-Gaussian phenomena: Material mechanics

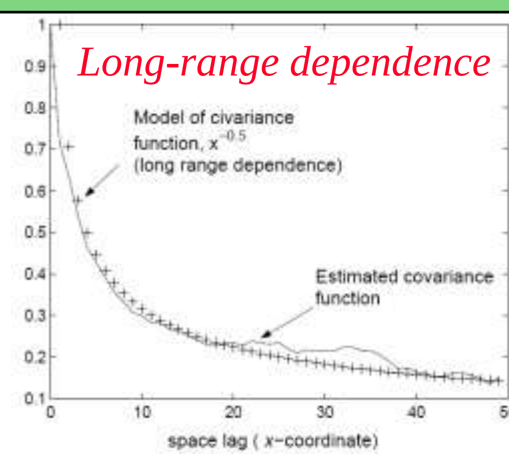
Polycrystals in aluminum





Crack trajectories





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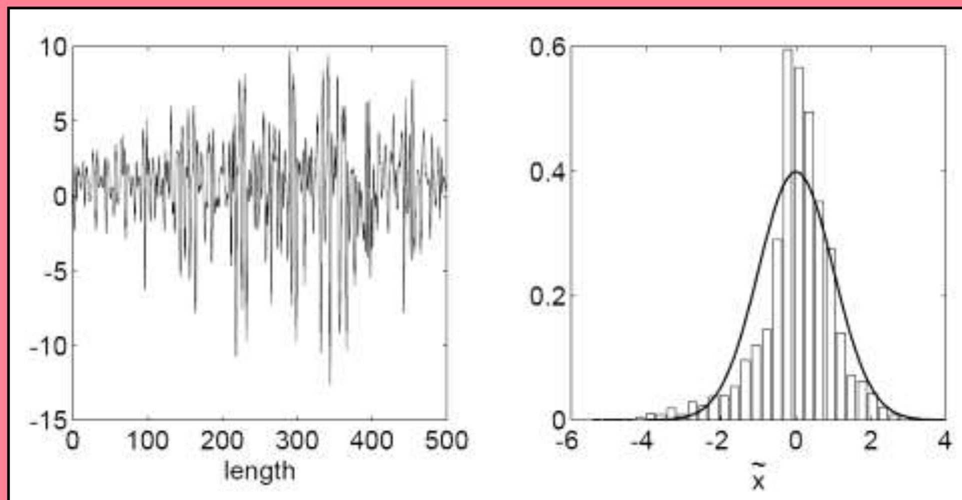
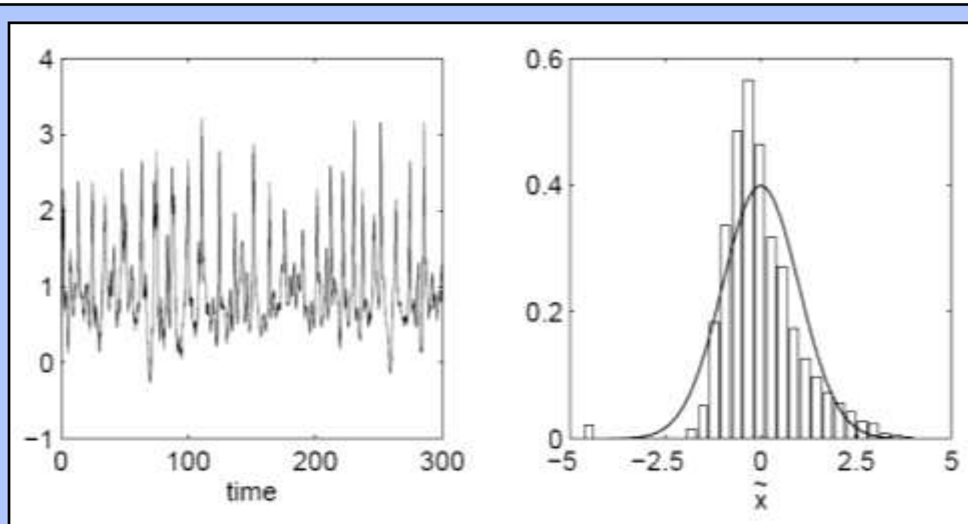
Example of non-Gaussian phenomena: Dynamic excitation



Coastal wave elevation (m) at Duck, NC in 1980



Railway track irregularities (mm) for 500 meter track

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Estimation of Foam Mechanical Properties

- Foam mechanical properties and variation necessary for model validation and uncertainty quantification studies
- Limited number of foam samples available
- Structural dynamics model to be developed
  - Linear elastic (density, elastic modulus, Poisson's ratio)
  - No Failure modeling
- Analysis will be done using
  - Optimization
  - Bayesian updating

Experiment Description

- Foam samples 1.5 inches by 0.5 inches thick
- Nine total samples
- A steel ball bearing dropped on sample
- Acoustic emissions measured and analyzed for natural frequencies
- Density estimated by measuring dimensions of sample and weighing.

Optimization

Sample Number	Elastic Modulus (E) (psi)	Poisson's Ratio (nu)
1	120,600	0.192
2	112,780	0.201
3	116090	0.199
4	103,080	0.201
5	117,940	0.198
6	109,850	0.200
7	107,630	0.197
8	109,710	0.204
9	111,190	0.197
Mean	112,100	0.199

$$e = \sum_{j=1}^{3} \left(\frac{f_i^{\,j} - f_{calc}^{\,j}}{f_{calc}^{\,j}} \right)^2$$

- Cost Function a least squares fit to the data
- Find E, nu such that error is minimum
- Fit data from each sample individually
- Frequencies are correlated

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Bayesian Updating

Likelihood

$$p(\theta|\varepsilon) \propto p(\varepsilon|\theta)p(\theta)$$

Posterior Distribution

Prior Distribution

- θ are the variables of interest
  - Elastic Modulus
  - Poisson's ratio
  - Statistical spread of error
- ε is the normalized frequency error

- Difficult to solve analytically except for a few special distributions
- Typically solved using Markov Chain Monte Carlo (MCMC)
- Some form of surrogate model typically necessary

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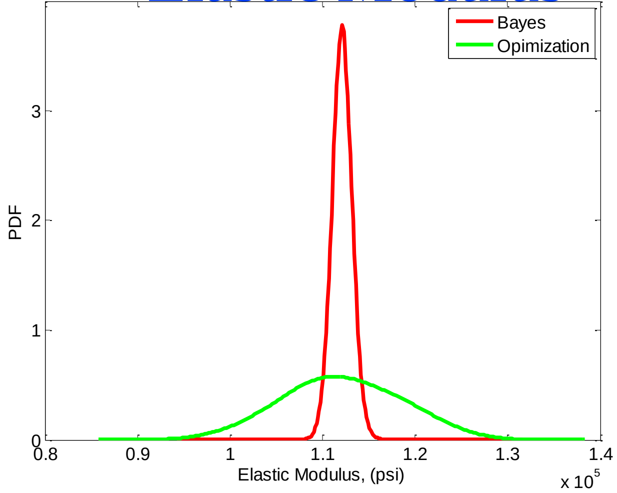
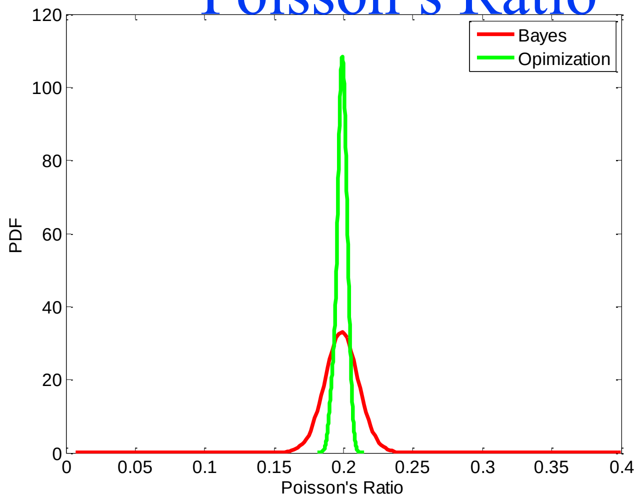
Comparison of Results

Elastic Modulus



Poisson's Ratio



- Different techniques give different distributions
- Important to describe assumptions and formulations
- Techniques exist to assess which model is more appropriate
- Both are correct within their assumptions

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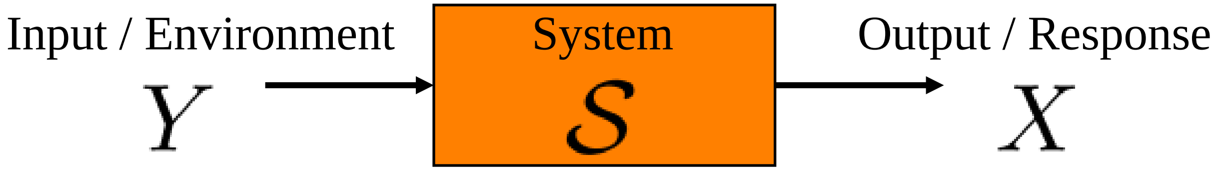
Methods for uncertainty propagation



Given representations for uncertainty in *Y* and/or *S*, how do we propagate this information to *X* ?

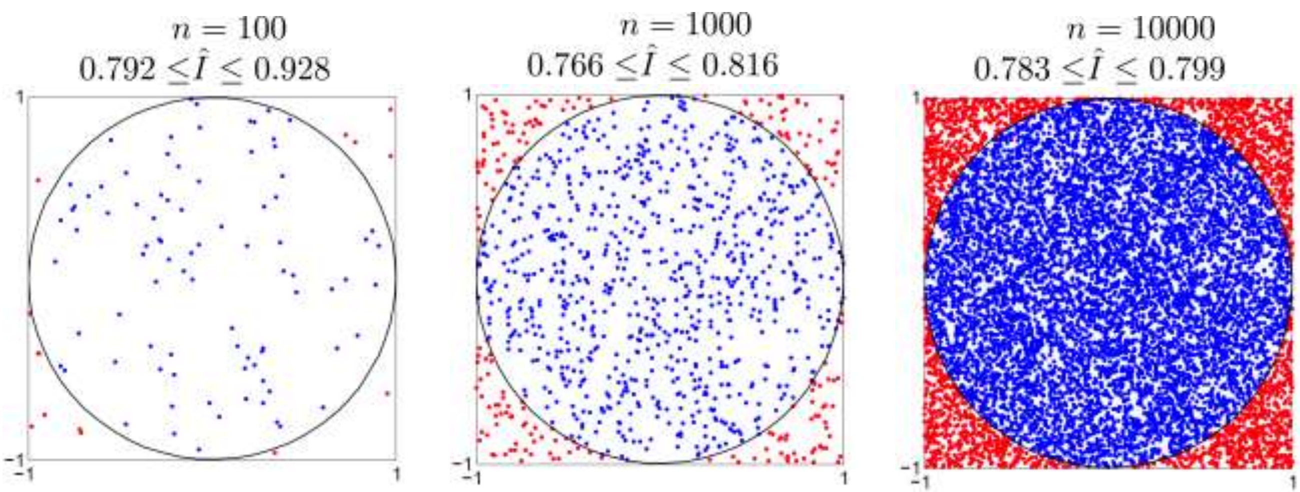
- Sampling-based methods (Monte Carlo, LHS)
  - General, simple, robust
  - Easily used with deterministic FE codes
  - Error estimates provided (confidence intervals)
  - Require many samples to estimate small probabilities
- Reliability-based methods (AMV, AMV+, FORM, SORM)
  - Can estimate small probabilities with far fewer samples
  - Assumptions may not be valid (*i.e.* linearity of failure surface)
  - Can be inaccurate (may diverge) and no error estimates provided
  - In general, limited to single failure mode or scalar response

More on Monte Carlo simulation

- Tool for numerical integration
  - Most quantities of interest in uncertainty analysis can be expressed as integrals

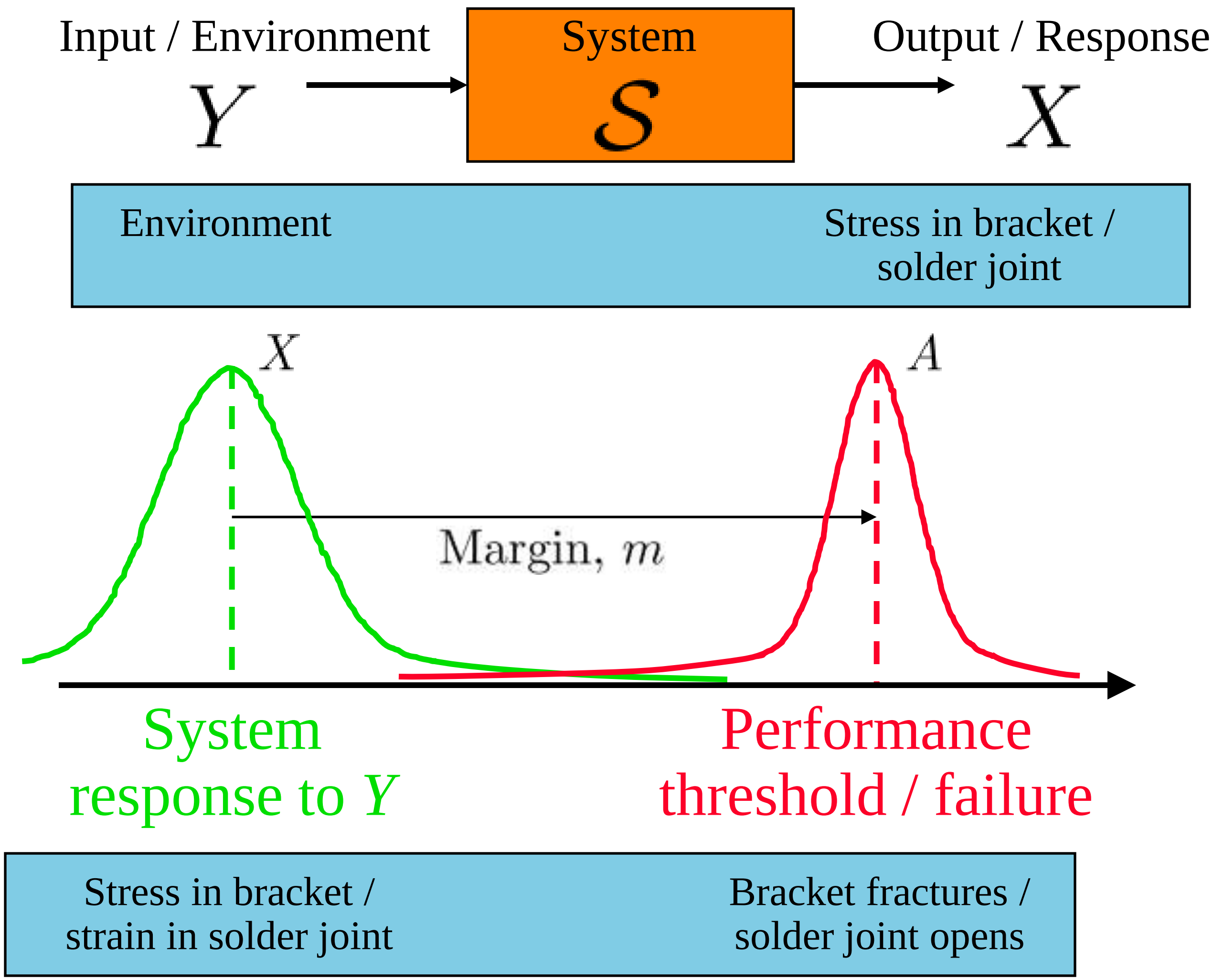
$$\mathrm{E}[X^p] = \int_{-\infty}^{\infty} x^p \, f(x) \, \mathrm{d}x \qquad\qquad \Pr(X \in \mathcal{A}) = \int_{\mathcal{A}} f(x) \, \mathrm{d}x$$

  - Developed at LANL in the 1940s
- Example: ratio of area of square to area of inscribed circle



$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \mathrm{d}y \, \mathrm{d}x = \frac{\pi}{4} \approx 0.785$$

Margin concepts



- Objective: determine if
  - $m \gg 0$ (good)
  - $m > 0$ (fair)
  - $m \leq 0$ (poor)
  - – Understand how *m* changes in time
- Uncertainty in system response and threshold / failure definition

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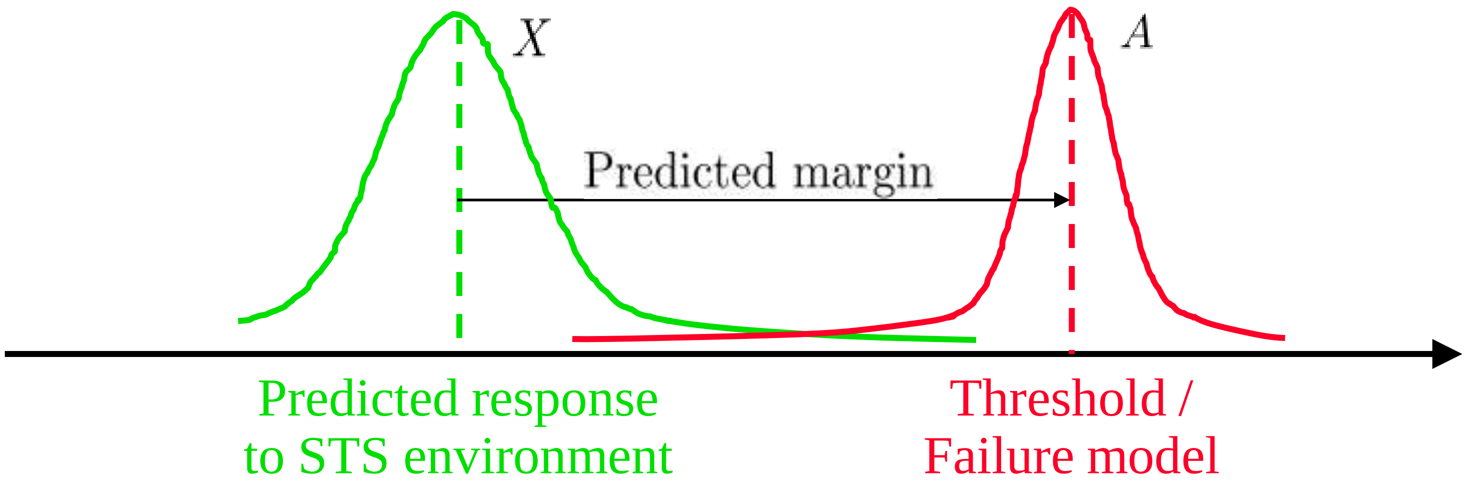
Margin assessment: combined modeling and testing approach



- Response predictions
  - Use validated system models that include uncertainty
  - Uncertainty models must be also be validated with experimental data
- Failure models
  - Failure is not well-understood for many problems, but it is a crucial element in predicting margin
  - Modeling & Simulation can have a big impact here

Summary

- Uncertainty modeling and analysis is an essential part of engineering design and analysis
  - Examples of uncertainty in complex systems
  - Use of probability theory to represent uncertainty
  - Methods and tools for uncertainty propagation
  - Quantification of Margins and Uncertainty